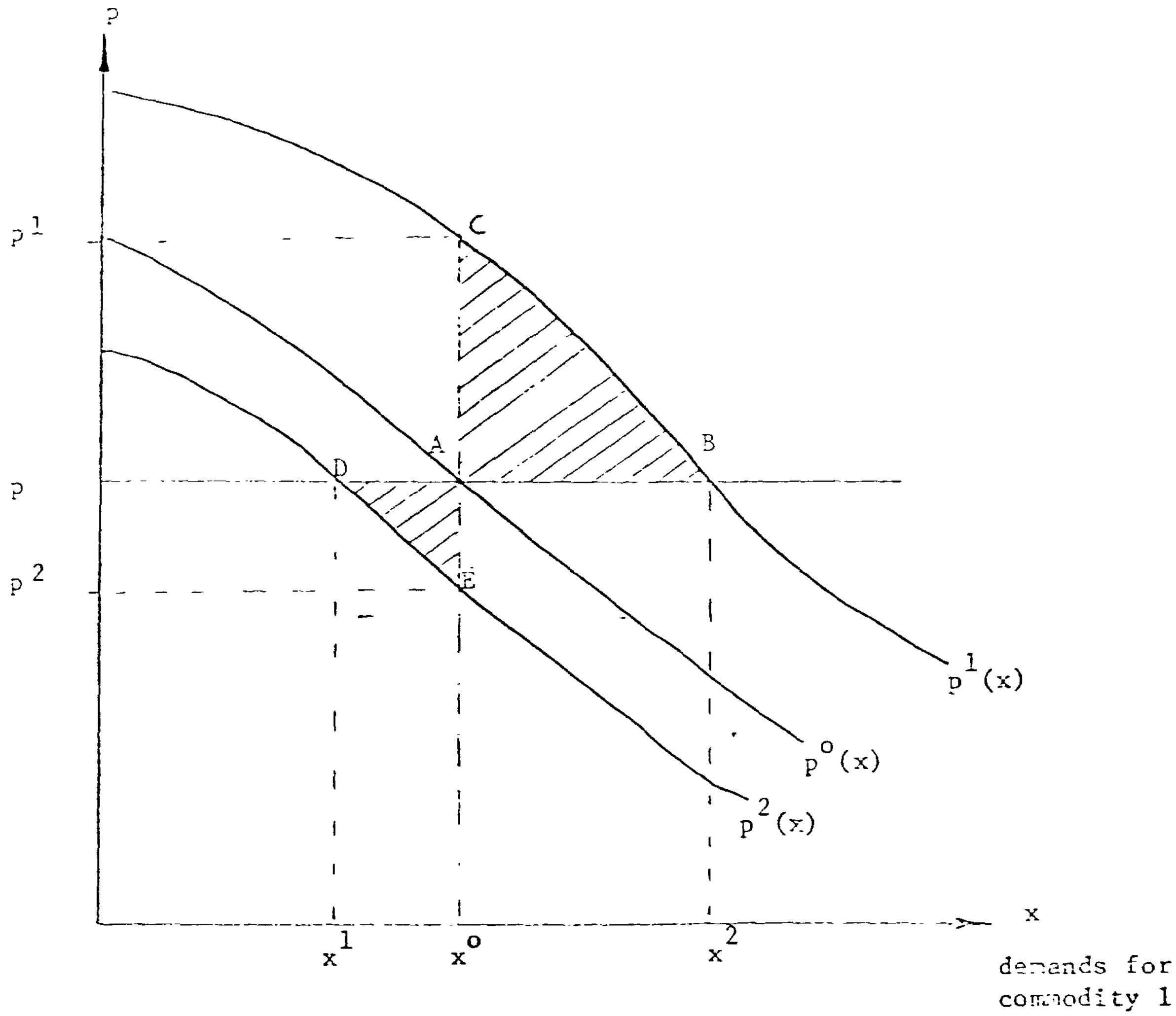
Figure 3.1

The Value of Information



$\mathrm{var}\{p^s(x^o)\}$ is the variance of full information demand prices for the quality of x pruchased with imperfect information. $\frac{dp^o(x^o)}{dx}$ is the steepness of the incomplete information, inverse demand curve. The value of information is therefore an increasing function of the dispersion of demand prices and of the price sensitivity of demand.

3.3 Logarithmic Utility Functions

We now begin the process of relaxing the strong assumption of constant marginal utility. First we consider the issues for the special case in which the utility function takes on the simple form:

$$U = \sum_{i=1}^{n} \Theta_i \ln\beta_i x_i, \qquad \Theta_i,\ \beta_i \geq 0$$

where $\Theta = \Theta(s)$ and $\beta = \beta(s)$

In the absence of further information about the true state the consumer chooses a consumption bundle x^o yielding the solution of:

$$\mathrm{Max}\{\underset{s}{E}u(x;s)\,|\,p'x \leq M\}$$

Note first that we can rewrite U as

$$U = \sum_i \Theta_i \ln\beta_i + \sum_i \Theta_i \ln x_i$$

Therefore x^o is the solution of

$$\mathrm{Max}\{\underset{s}{E}\Sigma\Theta_i \ln x_i\,|\,p'x \leq M\}$$

It follows that information leading to a change in beliefs about the vector β but not α has no effect upon the optimal consumption bundle. In particular suppose the only uncertain parameter is β_1. For example a consumer might be uncertain about the quality per unit of a particular commodity. Then for the logarithmic case information about the true value of β has no effect upon the optimal consumption bundle x^o. Moreover the knowledge that β_1 will be known prior to the time of purchase has no effect upon the ex ante utility level. That is, the value of perfect information about β_1 is zero.

To generate a model in which information changes actions we therefore focus upon cases in which the vector $\Theta = (\Theta_1,\ldots,\Theta_n)$ is uncertain. Without further loss of generality we may set $\beta = (1,1,\ldots,1)$.

Consider the case in which

$$\Theta_1 = s$$

$$\Theta_i = (1-s)\gamma_i \qquad i = 2,\ldots,n$$

$$\text{where } \sum_{2}^{n} \gamma_i = 1$$

Such a consumer is uncertain about his marginal valuation of commodity 1 <u>relative</u> to all other commodities but always spends his income on commodities 2,...,n in the same proportion. Given constant prices we may apply Hick's aggregation theorem and write the objective as

$$\underset{s}{\mathrm{Max}}\{E(s\ln x_1 + (1 - s)\ln y)|p_1x_1 + y = M\} \tag{3.12}$$

In the absence of further information about the true state this problem reduces to the certainty equivalent problem:

$$\underset{x_1,y}{\mathrm{Max}}\ \{\bar{s}\ln x_1 + (1 - \bar{s})\ln y|p_1x_1 + y \leq M\} \tag{3.13}$$

Solving we have:

$$U^o = \bar{s}\ln\bar{s} + (1 - \bar{s})\ln(1 - \bar{s}) - \bar{s}\ln p_1 + \ln M \tag{3.14}$$

Having paid V for perfect information about the true state the consumer chooses x(s) to yield the solution of:

$$\underset{x_1,y}{\mathrm{Max}}\{s\ln x_1 + (1 - s)\ln y|p_1x_1 + y \leq M - V\}$$

Since this problem has exactly the form of problem (3.13) the solution u(s) takes the form of (3.14). We have

$$u(s) = s\ln s + (1 - s)\ln(1 - s) - s\ln p_1 + \ln(M - V)$$

Then the expected utility with full information prior to purchase is:

$$U^* = \underset{s}{E}\{s\ln s + (1 - s)\} - \bar{s}\ln p_1 + \ln(M - V) \tag{3.15}$$

The value of information V* is then the level of V such that U^o and U* are equal. Equating (3.14) and (3.15) and rearranging we have:

$$-\ln(1 - \frac{V^*}{M}) = \underset{s}{E}[s\ln s + (1 - s)\ln(1 - s)] - [\bar{s}\ln\bar{s} + (1 - \bar{s})\ln(1 - \bar{s})] \tag{3.16}$$

The first bracketed term is a strictly concave function and the second term is the value of this function at $\bar{s}$, the mean level of s. Then by Jensen's inequality this expression is necessarily positive. Expanding both sides using Taylor's approximation we also have,

$$\frac{V^*}{M} \approx \frac{1}{2}[\frac{1}{s} + \frac{1}{(1 - s)}]\mathrm{var}(s) \tag{3.17}$$

$$= \mathrm{var}(s)/2(1 - \bar{s})\bar{s}$$

It is interesting to compare this with the 'consumer surplus' estimate of the previous section. For the logarithmic utility function:

$$p^s(x) = sM/x$$

Substituting into (3.13) the Marshallian approximation can be written as

$$M\mathrm{var}(s)2\bar{s}$$

Comparing this with (3.17) it follows that the Marshallian estimate of the value of perfect information is biassed downwards by a factor of (1 - $\bar{s}$). The two estimates differ because in the logarithmic case a change in s changes not only the demand curves for x_1 but also the damand for other goods y. When the triangles corresponding to those in Figure 1 are computed for both x_1 and y and the average areas are added together the resulting estimate of V* is indeed (3.17). All this suggests that the average area calculation is capable of further generalization. In Section 3.4 we shall see that this is indeed the case.

We conclude this section with a comparison of the exact value of information given by equation (3.16), with the approximation given by equation (3.17). Suppose s takes on two values $\bar{s} + \varepsilon$ and $\bar{s} - \varepsilon$ with equal probability.

Let

$$-\ln(1 - \frac{V^*}{M}) = A$$

Then

$$V^* = M(1 - e^{-A})$$

where from (3.16).

$$A = \frac{1}{2}[(\bar{s}+\varepsilon)\ln(1+\frac{\varepsilon}{\bar{s}})+(\bar{s}-\varepsilon)\ln(1-\frac{\varepsilon}{s})+(1-\bar{s}-\varepsilon)\ln(1-\frac{\varepsilon}{1-\bar{s}})+(1-\bar{s}-\varepsilon)\ln(1+\frac{\varepsilon}{1-\bar{s}})]$$

Also from (3.17) the approximation to the value of information can be expressed as:

$$V^*_a = \frac{M}{2}\varepsilon^2(\frac{1}{\bar{s}} + \frac{1}{1 - \bar{s}}).$$

Computational results are summarized in the following tables. Note that $V^*(\bar{s}) = V^*(1-\bar{s})$ and $V^*_a = V^*_a(1-\bar{s})$. Therefore the value of information for $\bar{s}$ = .7, .9 .99 can also be obtained from the two tables.

Comparison of these tables indicates that the approximation is remarkably good over the whole range of feasible values of $\bar{s}$. For example the mean difference between the ten computed values of V^*_a and V* expressed as a percentage of V*, is less than 6.5%. This is reason for having some confidence that the results developed in the next sections tield reasonably good approximations of V*.

3.4 General Utility Functions

We now consider the value of perfect information for any utility function u(x;s) which is twice differentiable in x and s and strictly quasi-concave in x. In contrast to the above discussion we allow not only x but also s to be a vector.

Suppose first that perfect information is provided at no cost. Then the consumer chooses x(p;s) yielding the solution of:

$$u(s) = \text{Max}\{u(x,s)|p'x \leq M\} \qquad (3.18)$$

The expected utility thereby achieved is:

$$u^*(M) = \underset{s}{E}u(x(p,s);s)$$

Without the information the consumer chooses x^o to achieve an expected utility of:

$$u^o(M) = \underset{x}{\text{Max}}\{Eu(x,s)|p'x \leq M\}$$

Table 3.1

The Value of Perfect Information as a Percentage of Income

ε \ $\bar{s}$	.01	.10	.30	.50
.01	.696	.056	.024	.020
.10		7.215	2.387	1.994
.30			23.994	17.532
.50				50.003

Table 3.2

Approximation of the Value of Perfect Information
as a Percentage of Income

ε \ $\bar{s}$	.01	.10	.30	.50
.01	.505	.056	.024	.020
.10		5.556	2.417	2.000
.30			21.750	18.000
.50				50.000

Let s^o be that value of s so that:

$$x^o = x(p,s^o)$$

Then the increase in utility associated with having perfect information is:

$$u^*(M) - u^o(M) = \underset{s}{E}[u(x(p,s);s) - u(x(p,s^o);s)]$$

Expanding the right hand side according to Taylor's approximation we have

$$u^*(M) - u^o(M) = \underset{s}{E}[(x-x^o)'u_x + \frac{1}{2}(x-x^o)'u_{xx}(x-x^o) + (x-x^o)'u_{xs}(s-s^o)+\ldots] \tag{3.19}$$

Since x(p,s) is the solution of (18) it must satisfy the Kuhn-Tucker necessary conditions for the following Lagrangian:

$$L(x;\lambda;s)= u(x;s) + \lambda(m-p'x)$$

Assuming that x(p,s) is an interior solution we have:

$$L_x = u_x(x(p,s);s) - \lambda p=0 \tag{3.20}$$

Then the first term inside the bracket of expression(3.19)reduces to:

$$(x-x^o)'\lambda p = \lambda p'(x-x^o) = \lambda(p'x-p'x^o) = 0$$

Moreover, differentiating(3.20)with respect to both s and p we have:

$$u_{xx}x_p = \lambda I + p\lambda_p' \tag{3.21}$$

and

$$u_{xx}x_s + u_{xs} = p\lambda_s' \tag{3.22}$$

Linearizing the demand curves x(p;s) we have:

$$(x-x^o) \approx \mathbf{x}_s(s-s^o) \tag{3.23}$$

Prior to the receipt of information x(p,s) is a random variable. Then actual demand $\tilde{x}$, can be thought of as a random drawing from the set $X = \{x|x = x(p,s);s \in S\}$. The Marshallian demand price vector associated with consumption vector x^o is therefore:

$$\tilde{p} = \{p:x(\tilde{p},s) = x^o\}$$

Then

$$\tilde{x} - x^o \approx x_p(\tilde{p}-p) \tag{3.24}$$

Utilizing (3.22)we can rewrite the third term in the bracket of (3.19)as follow:

$$(x-x^o)'u_{xs}(s-s^o) = (x-x^o)'(p\lambda_s' - u_{xx}x_s)(s-s^o)$$

$$= (x-x^o)'p\lambda_s' (s-s^o) - (x-x^o)'u_{xx}x_s(s-s^o)$$

The first term on the right hand side is zero since $p'x = p'x^o$. Then using the linear approximation (3.23)we have:

$$(x-x^o)'u_{xs}(s-s^o) \simeq -(x-x^o)u_{xx}(x-x^o)$$

The increase in utility associated with having perfect information can therefore he approximated as follows:

$$u^*(M) - u^o(M) \simeq \frac{1}{2}\underset{s}{E}(\tilde{x}-x^o)'u_{xs}(s-s^o)$$

$$\simeq -\frac{1}{2}E(\tilde{x}-x^o)'u_{xx}(\tilde{x}-x^o) \qquad (3.25)$$

Substituting for u_{xx} and $x-x^o$ from (21) and (24) we have:

$$u^*(M) - u^o(M) \simeq -\frac{1}{2}E(x-x^o)'(\lambda I+p\lambda_p')(\tilde{p}-p)$$

$$= -\frac{1}{2}E\{(\tilde{p}-p)'(\tilde{x}-x^o)\lambda\}$$

From the first order conditions we have:

$$\frac{\partial u^*(M)}{\partial M} = \underset{s}{E}\lambda(s)$$

Therefore, ignoring the impact of variation across states in the marginal utility of income we have:

$$u^*(M) - u^o(M) = -\frac{1}{2}\frac{\partial u^*}{\partial M}E\{(\tilde{p}-p)'x_p(\tilde{p}-p)\} \qquad (3.26)$$

For the final step we note that the value of information is that level V* such that:

$$u^*(M-V^*) = u^o(M).$$

Taking first order approximation about V* = 0 we have:

$$u^*(M - V^*) \simeq u^*(M) - \frac{\partial u^*(M)}{\partial M}V^* \qquad (3.27)$$

Comparing (3.26)and (3.27)it follows that:

$$V^* \simeq -\frac{1}{2}E(\tilde{p}-p)'x_p(\tilde{p}-p) \qquad (3.28)$$

Suppose only the demand price of commodity 1 varies with s. Then:

$$V^* \simeq \frac{1}{2}\left|\frac{\partial x_1}{\partial p}\right|\mathrm{var}(\tilde{p}_1-p_1)$$

Comparing this with expression (11) it follows that our approximation does correspond to that obtained in Section 3.2.

Similarly, for the logarithmic utility functions it is a straightforward exercise to show the approximation given (3.28) reduces to the expression obtained in Section 3.3.

3.5 The Value of Imperfect Information

The preceding sections were concerned with valuing information which eliminated all uncertainty about the effects of consuming various goods. V* represented what the consumer would pay for perfect information about s. But it is seldom feasible for research to eliminate all uncertainty about the characteristics of goods. Realistically, investigation only narrows the range in which the true characteristics lie, decreasing but not eliminating the dispersion of the consumer's probability distribution over s. In this section we ask how much a consumer would be willing to pay for such imperfect information.

The outcome of the research the consumer commissions, or message he receives, will be denoted by $\alpha \varepsilon A$ where A is the set of possible results. Before the research is conducted a is a random variable in the mind of the consumer. Its relation to the uncertain state of the world is embodied in a subjective joint probability distribution function $F(\alpha,s)$ over A x S; F(s), $F(\alpha)$, $F(s|\alpha)$ denote the associated marginal and conditional probability distributions. This pair [A, $F(\alpha,s)$] is the information structure whose value we wish to determine.

If the information is provided at no cost, and if only s not the message itself affects his ultimate welfare, then upon receiving α the consumer chooses $x(p,\alpha) \equiv x$ to obtain conditional level of expected utility

$$E_{s|\alpha}\, u(x;s) \;=\; \underset{x}{\text{Max}}\{E_{s|\alpha}u(x;s)\,|\; p'\,x \le \tag{3.29}$$

Prior *to* the receipt of *a,* x is a random variable, given α it is no longer random even though s may still be unknown. The anticipated level of expected utility prior to receipt of the message, depending both on the information structure and income, is:

$$u^*(M;A) \;=\; E_{s,\alpha}\, u(\tilde{x};s) \;=\; E_\alpha E_{s|\alpha}u(\tilde{x};s). \tag{3.30}$$

As before, the consumer chooses x^o without the information to achieve an expected utility of:

$$u^o(M) \;=\; \underset{x}{\text{Max}}\; \{E_s u(x;s)\,|\; p'x \le M\}$$

and the increase in expected utility associated with having the information structure is:

$$u^*(M;A) - u^o(M) = E_\alpha E_s|_\alpha\; [u(\tilde{x};s) - u(x^o;s)]. \tag{3.31}$$

Expanding the inner expectation of the right hand side in a Taylor series in x around $\tilde{x}$ yields:

$$\begin{aligned} E_{s|\alpha}\; [u(\tilde{x};s) - u(x^o;s)] \;&=\; -E_{s|\alpha}\; [u(x^o;s) - u(\tilde{x};s)] \\ &= -E_{s|\alpha}\; [\; (x^o - \tilde{x})'u_x + 1/2\; (x^o - \tilde{x})'u_{xx}\; (x^o - \tilde{x}) + . \end{aligned} \tag{3.32}$$

Recalling that $\tilde{x}$ was the solution to (29), and forming the Langrangian

$$L(x,\lambda;a) \;=\; E_{s|\alpha}u(x;s) \;+\; \lambda(M-p'x),$$

x must satisfy the first order condition:

$$L_x \;=\; E_{s|\alpha}\; u_x(\tilde{x};s) \;-\; \tilde{\lambda}p \;=\; 0. \tag{3.33}$$

The scalar λ denotes the expected marginal utility of income conditional on research outcome α being received. Differentiating

with respect to p provides the additional relation (3.33)

$$[E_{s|\alpha}\, u_{xx}]\tilde{x}_p = \tilde{\lambda} I + p\tilde{\lambda}_p'. \tag{3.34}$$

Note that $\tilde{\lambda}$, x_p, $\tilde{x}$, $\tilde{\lambda}p$ are non-random once α is revealed. Substituting (3.33) into the first component of the right hand side of(3.32) tells us that:

$$E_{s|\alpha}\,(x^o-\tilde{x})'u_x = (x^o-\tilde{x})'E_{s|\alpha}u_x = (x^o-\tilde{x})'p\tilde{\lambda} = 0$$

since $x^o\, p - \tilde{x}'p = M$ from the budget constraints.

Hence(3.31) is approximated by:

$$u^*(M;A) - u^o(M) \simeq E_\alpha E_{s|\alpha}\,[-1/2\,(x^o-\tilde{x})'u_{xx}(x^o-\tilde{x})] \tag{3.35}$$

$$= -1/2\; E_\alpha\,(x^o-\tilde{x})'\,[E_{s|\alpha}u_{xx}](x^o-\tilde{x})$$

Now define the Marshallian demand price vector $\tilde{p}$ associated with the consumption vector x^o conditional on message a being received as:

$$\tilde{p} = \{\tilde{p}:\; x(\tilde{p},\alpha) = x^o\}.$$

Linearly approximating the demand function for given α around p gives

$$(x^o-\tilde{x}) \simeq \tilde{x}_p\;(\tilde{p}-p). \tag{3.36}$$

Substituting(3.36)into the right hand side of(3.35)yields:

$$-1/2\; E_\alpha(x^o-\tilde{x})'\;[E_{s|\alpha}u_{xx}]\tilde{x}_p(\tilde{p}-p)$$

which can be written utilizing relation (34) as:

$$-1/2\; E_\alpha\;(x^o-\tilde{x})'\;[\tilde{\lambda}I + p\tilde{\lambda}_p{}']\;(\tilde{p}-p).$$

The $(x^o-\tilde{x})'p\tilde{\lambda}_p'(\tilde{p}-p)$ portion of their expression vanishes since $(x^o-\tilde{x})'p = p'x^o - p'\tilde{x} = M-M = 0$ from the budget constraints. Using(3.36)again on the remaining portion of the expression results in:

$$u^*(M;A)-u^o(M) \simeq -1/2\; E_\alpha\tilde{\lambda}(\tilde{p}-p)'\tilde{x}_p(\tilde{p}-p). \tag{3.37}$$

Prior to receipt of the message the expected marginal utility of income is

$$\frac{\partial u^*(M;A)}{\partial M} = E_\alpha\tilde{\lambda}.$$

If the effect of messages on the slopes $\tilde{x}_p$ of the demand curves is negligible, and if we ignore any correlation between $\tilde{\lambda}$ and the remaining quadratic form in(3.37),then the expected gain in utility may be written almost precisely as in (3.26):

$$u^*(M;A)-u^o(M) = -1/2\,\frac{\partial u^*}{\partial M}\; E_\alpha[(\tilde{p}-p)'x_p(\tilde{p}-p)]. \tag{3.38}$$

Analagously defining the value of the information structure as V^*_A for which:

$$u^*(M-V^*_A;A) = u^o(M),$$

one obtains a first order approximation to V* of

$$V^* \approx -1/2\ E_\alpha(\tilde{p}-p)' x_p (\tilde{p}-p). \tag{3.39}$$

Although it is an approximation, (3.39) provides a consistent estimate of the value of improving a consumer's estimate of s over a wide range of information structures. For example, if the research will provide perfect information, as when A coincides with S and $\alpha = s$, then (3.39) is identical to (3.28). If the research outcome in fact sheds no light on s, so that $x(p,\alpha) = x^o$ for all outcomes, then $\tilde{p} = p$ for all α and (3.39) indicates $V^*_A = 0$. More importantly, (3.39) makes it clear that research whose results would not change consumers' behaviour is valueless, even though it may significantly improve estimates of s in a purely statistical sense.

One final check on the plausibility of (3.39) as an approximate indicator of the value of imperfect information about the consequences of consuming various goods is to verify that information never has a negative value. Such a result must follow if the outcome of the research itself, as opposed to the true characteristics of goods s, has no direct effect on the consumer's utility. That (3.39) has this property can be demonstrated as follows. Assuming as we have that the slopes of the uncompensated demand curves as indicated by $x_p \equiv [\partial x_i/\partial p_j]$ are unaffected by the outcome of the research a, these slopes will be identical to those of the demand curves if no information was to be received. Using the Slutsky relation of conventional demand theory

$$\partial x_i/\partial p_j = \partial x_i^c/\partial p_j - x_j \partial x_i/\partial M$$

in which $x_j^c/\partial p_j$ is the slope of the income-compensated demand curve for good i with respect to the price of good j, we can express x_p as $x_p^c - x_m x'$ in which $x \equiv x^o$ is the consumption point at which the derivatives are evaluated. Inserting this expression for x_p into (3.39) gives us the alternate form

$$V^* \approx -1/2\ E_\alpha(\tilde{p}-p)'[x_p^c - x_M x^{o'}]\ (\tilde{p}-p).$$

But since $p'x^o = \tilde{p}'x^o = M$ from the budget constraints and definition of $\tilde{p}$, the second component of the inner bracketed expression becomes 0 when multiplied by (p-p). Thus (3.39) can be alternately written as

$$V^* \approx -1/2\ E_\alpha(\tilde{p}-p)' x_p^c (\tilde{p}-p). \tag{3.40}$$

The Stutsky matrix x_p^c is known to be symmetric and negative semidefinite. Hence the expectation of the quadratic form in (3.40) is non-positive and V* must be non-negative for all information structures.

3.6 Information and Price Adjustment

As analyzed in Section 3.1 of this report, information is valuable to the extent that consumption plans change with the message received. Loosely, the greater the optimal adjustment to the different messages the more an individual is willing to pay ex-ante for the provision of the information. Ignored, however, is the possibility that the receipt of information will have significant price effects.

Implicity in such a formulation is the assumption that prices are largely determined by cost conditions rather than the intersection of supply and demand curves. While this is a natural first approximation for a variety of applications it is particularly inappropriate for non-produced commodities of uncertain quality. One important case is the adjustment of land prices to reflect differences in air quality in an urban environment. It is this case that we shall focus on in the following sections.

We begin in Section 3.7 by illustrating the implications of price adjustment on the value of information for a simple exchange economy. It is shown that <u>all</u> agents in an economy may be made worse off by the announcement that the true quality of a product will be made known prior to trading. Essentially the anticipation of information introduces an additional distributive risk which reduces each individual's expected utility. It is shown that each agent would prefer to engage in a round of trading prior to the revelation of product quality, thereby insuring himself against an undesirable outcome.

The in Section 3.8 a simple urban model is developed in which a fixed number of individuals must be located in two regions. The equilibrium allocation of individuals is first examined. Simple sufficient conditions for higher income groups to locate in the preferred environment are established.

Surprisingly, it is shown that under non implausible alternative conditions <u>both</u> tails of the income distribution may locate in the preferred environment.

Section 3.9 asks what allocation of land and goods maximize a symmetric social welfare function. Starting with income equally distributed it is shown that optimization in general requires an income transfer from those living in one zone to those in the other. Under the conditions which imply that in equilibrium the rich will locate in the better environment, it is optimal to transfer income to those in the better environment from the remainder of the population! The intuition behind this paradoxical conclusion is then developed.

Finally, Section 3.10 focusses on the implications of conducting research to resolve uncertainty about the nature of the environmental hazard.

<u>3.7 Information About Product Quality with Negative Social Value</u>

Consider a two person economy in which aggregate endowments of two commodities, X and Y, are fixed and equal to unity. Both individuals have utility functions of the form:

$$u(x_i, y_i; \Theta) = (\Theta x_i)^{1/2} + y_i^{1/2} \quad i = 1,2$$

where Θ is a parameter reflecting the 'quality' of the product. Prior to trading Θ is unknown but both individuals believe that with equal probability Θ takes on the values 0 and 1.

Then the expected utility of agent i is:

$$U^\circ(x_i,y_i) - \underset{\Theta}{E}u(x_i,y_i;\Theta) = \frac{1}{2}x_i^{1/2} + y_i^{1/2} \tag{3.41}$$

Without loss of generality we may set the price of y equal to unity. Then each agent chooses (x_i,y_i) to maximize U° subject to a budget constraint

$$px_i + y_i \leq p\bar{x}_i + \bar{y}_i$$

where $(\bar{x}_i,\bar{y}_i)$ is the agent's endowment.

Since U° is strictly concave the following first order condition yields the global maximum.

$$\frac{\partial U^\circ}{\partial x_i} \Big/ \frac{\partial U^\circ}{\partial y_i} = \frac{1}{4}x_i^{-1/2} \Big/ \frac{1}{2}y_i^{-1/2} = \frac{1}{2}\left(\frac{y_i}{x_i}\right)^{1/2} = p$$

Then:

$$\frac{y_i}{x_i} = 4p^2 \qquad i = 1,2 \tag{3.42}$$

It follows that:

$$1 = \frac{\Sigma y_i}{\Sigma x_i} = 4p^2$$

Thus the equilibrium price of x is 1/2 and from(3.42) $y_i = x_i$, $i = 1,2$.

Suppose $(\bar{x}_1,\bar{y}_1) = (1,0)$ and $(\bar{x}_2,\bar{y}_2) = (0,1)$. Then from the budget constraint it is a straightforward matter to show that:

$$(x_1,y_1) = (1/3,1/3) \text{ and } (x_2,y_2) = (2/3,2/3)$$

From(3.41)theexpected utility of the agents is given by:

$$U_1^\circ = \frac{3}{2}^{1/2} \qquad U_2^\circ = \left(\frac{3}{2}\right)^{1/2}$$

Next suppose that research is to be conducted which will reveal the true state prior to any trading. If $\Theta = 0$ the endowment of agent 1 is valueless hence there can be no trade ex post. Then:

$$u_1(\Theta=0) = 0 \quad \text{and} \quad u_2(\Theta=0) = 1$$

If $\Theta = 1$ each agent has an ex-post utility function:

$$u_i = x_i^{1/2} + y_i^{1/2}$$

Applying an almost identical argument to that made above, it can be shown that for such preferences the equilibrium price of x is unity and both agents consume half the aggregate endowment. Then:

$$u_1(\Theta=1) = 2^{1/2} = u_2(\Theta=1)$$

Prior to the revelation of the information both agents place an equal probability on the two possible states. Thus expected utility levels with the information are:

$$U_1^* = \frac{1}{2}u_1(\theta=0) + \frac{1}{2}u_1(\theta=1) = \left(\frac{1}{2}\right)^{1/2}$$

and

$$U_2^* = \frac{1}{2}u_2(\theta=0) + \frac{1}{2}u_2(\theta= 1) = \frac{1 + 2^{1/2}}{2}$$

Then $(U_1^*)^2 - (U_2^\circ)^2 = 1/2 - 3/4 < 0$

and $(U_2^*)^2 - (U_2^\circ)^2 = \frac{3 + 2\sqrt{2}}{4} - \frac{6}{4} < 0$

The prospect of information prior to trading therefore creates a distributive risk which reduces the expected utility of every agent!

Each agent would therefore like to insure himself against such risk. It follows that there are potential gains to opening the commodity market prior to the announcement of the true state. Since the future spot price of X relative to Y, $\hat{p}$, is independent of individual endowments it follows from the above analysis that $\hat{p} = 0$ if $\theta = 0$ and $\hat{p} = 1$ if $\theta - 1$, that is:

$$\hat{p}(\theta) = \theta; \ \theta = 0,1$$

If the spot price of X is p, agent i can select bundles (x_i, y_i) satisfying

$$px_i + y_i = p\bar{x}_i + \bar{y}_i \qquad (3.43)$$

When the state is announced the agent then makes a second round of exchanges subject to the contraint:

$$\hat{p}(\theta)x_i(\theta) + y_i(\theta) = \hat{p}(\theta)x_i + y_i \qquad \theta = 1,2. \qquad (3.44)$$

But if $\theta = 0$ the future spot price $\hat{p}(\theta) = 0$. It follows that there will be no trading after the announcement, that is:

$$(x_1(0), y_1(0)) = (x_i, y_i)$$

if $\theta = 1$ the future spot price, $\hat{p}(\theta) = 1$. Given the symmetry of the indifference curves each agent will trade in such a way as to equalize his spending on the two commodities.

Then $$(x_i(1), y_i(1)) = \left(\frac{x_i + y_i}{2}, \frac{x_i + y_i}{2}\right)^{1/2}$$

Expected utility of agent i is therefore

$$U(x_i, y_i) = \frac{1}{2}y_i^{1/2} + \left(\frac{x_i + y_i}{2}\right)^{1/2}$$

With a spot price of p, agent i chooses x_i and y_i to maximize $U(x_i, y_i)$ subject to his budget constraint(3.43). The first order condition for expected utility maximization is therefore:

$$\frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial y_i}} = \frac{\left(\frac{x_i}{y_i}+1\right)^{-1/2}}{2^{-1/2}+\left(\frac{x_i}{y_i}+1\right)^{-1/2}} \tag{3.45}$$

It follows that $\frac{x_i}{y_i}$ is the same for both agents, hence equal to $\frac{\Sigma x_i}{\Sigma y_i} = 1$. Then from (3.45) $p = 1/2$. From the budget constraint (3.43) it follows that

$$(x_1,y_1) = (1/3,1/3) \text{ and } (x_2,y_2) = (2/3,2/3)$$

But this is exactly the consumption achieved by each agent in the absence of the information. Therefore the prior trading just eliminates the undesired utility risk, and the expected value of the information is zero.[1/]

A central feature of this and the earlier results is that agents correctly anticipate the price implications of the state revealing message. If consumers are unaware of these implications the analysis of section 1 applies. Each will therefore place a positive value on the information.

Of course it is a long leap from this simple example to a general proposition. However it does seem reasonable that there will, in general, be a tendency for price adjustments to offset the anticipated gains associated with better information. Thus except in cases where there are solid ground for arguing that prices are cost determined, the expressions for the value of information developed in Section 3.1 seem likely to overstate true value.

3.8 Urban Location and Land Values with Environmental Hazards

One very important case in which price adjustments to changes in information are central, is that of urban location. To illustrate the issues we shall consider a city which consists of two zones.

The utility of any individual living in the second zone is a concave function $U(x,y)$ of the area of his residence x and expenditure on other commodities y. If provided the same bundle of commodities in the environmentally affected first zone his utility drops to $U(x,y)-s$. That is, s is the loss in utility associated with living in the "smoggy" first zone.

Suppose each individual purchases land from some outside landowner and all have identical incomes.[2/] Let P_i be the price of a unit of land in zone i. For those locating in the second zone the utility level achieved is:

$$V(p_2,I) = \underset{x,y}{\mathrm{Max}}\{U(x,y)\,|\,P_2x + y = I\} \tag{3.46}$$

Similarly for those locating in the first zone the utility level achieved is:

$$V(p_1, I) - s = \underset{x,y}{\mathrm{Max}}\{U(x,y)\,|\,p_1x + y = I\} \quad -s. \tag{3.47}$$

In the absence of constraints on land purchases, the value of land in the "smoggy" zone must fall until utility is equated in the two zones. This is depicted in Figure 3.2.

At the level of an individual consumer, one measure of the cost of the smog is the extra income H that a person living in the second zone would have to be given in order to make him willing to move at constant prices. In formal terms this is the Hicksian compensation required to maintain the utility level of an individual in the smoggy zone at the higher land value P_2, that is:

$$V(P_2,\ I + H) = V(P_1,\ I) = V(P_2,\ I) + s \tag{3.48}$$

This is also depicted in Figure 3.2.

With this background we can now ask which individuals live where, if incomes are not equally distributed. For expositional ease we shall restrict our attention to utility functions that are homothetic. Suppose that income is distributed continuously. Then for some income level I° individuals will be indifferent between living in the two zones. We therefore have:

$$V(P_2,\ I^\circ) = V(P_1,\ I^\circ) - s$$

An individual with income I > I° locates in the smog free zone if and only if:

$$V(P_2,\ I) > V(P_1,\ I) - s$$

Consider Figure 3.2. Those with incomes of I° are indifferent between C_1 and C_2 and hence between C_1' and C_2. Then:

$$V(P_2,\ I^\circ) = V(P_2,\ I^\circ + H^\circ) - s. \tag{3.49}$$

Moreover given our assumption that those with incomes of I locate in the smog free zone, they must prefer D_2 to D_1, and hence prefer D_2 to D_1'. Then:

$$V(P_2,\ I) > V(P_2,\ I + H) - s \tag{3.50}$$

Combining (3.49) and (3.50) the higher income group prefer zone 2 if and only if:

$$V(P_2,\ I + H) - V(P_2,\ I) < V(P_2,\ I^\circ + H^\circ) - V(P_2,\ I^\circ) \tag{3.51}$$

For the special case of homothetic preferences depicted in Figure 3.3 we also have:

$$\frac{OC_1'}{OC_1} = \frac{OD_1'}{OD_1}$$

Moreover, $$\frac{OD_1}{OC_1} = \frac{I}{I_\circ} \text{ and } \frac{OD_1'}{OC_1'} = \frac{I + H}{I_\circ + H_\circ}$$

Figure 3.2

Urban Location and Land Values

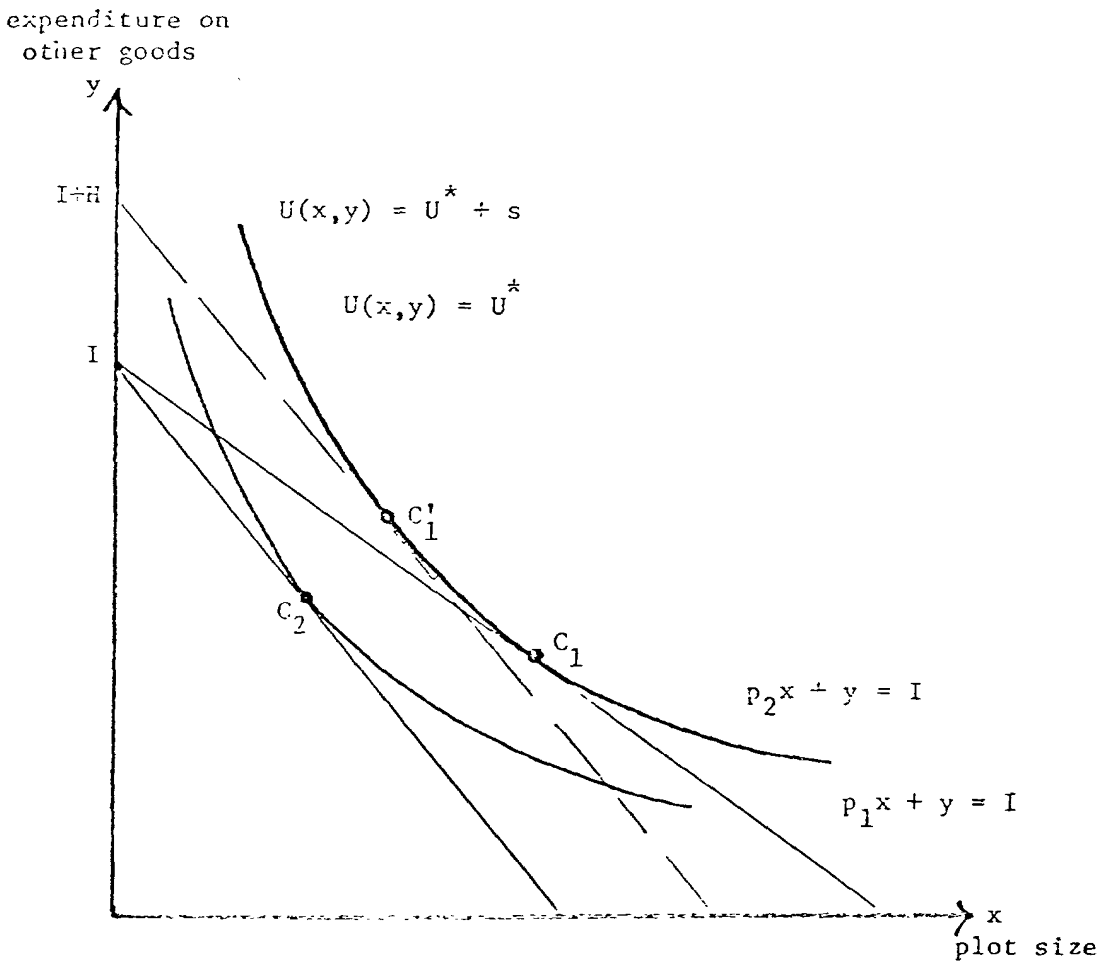
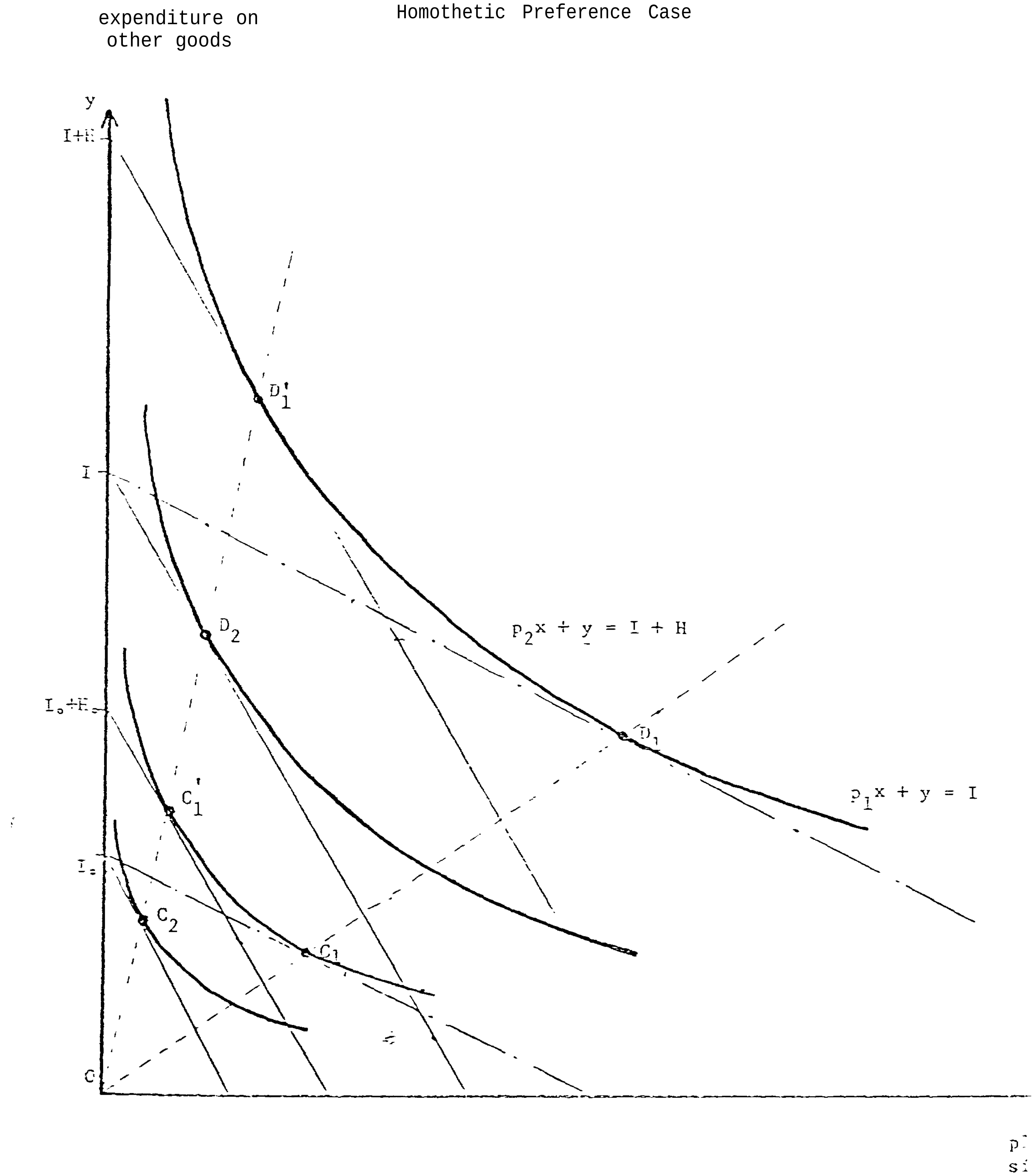


Figure 3.3

Homothetic Preference Case



It follows immediately that:

$$\frac{H_\circ}{I_\circ} = \frac{H}{I}$$

We may therefore rewrite the necessary and sufficient condition (3.51) as

$$V(P_2,(\frac{I_\circ + H_\circ}{I_\circ})I) - V(P_1, I) < V(P_2,I_\circ + H_\circ) - V(P_2,I_\circ) \qquad (3.51)'$$

Note that the left and right hand sides of (3.51)' are equal for $I = I_\circ$. Then a sufficient condition for all those with higher incomes to prefer zone 2 is that the left hand side of (3.51)' be decreasing in I, that is:

$$\frac{1}{I_\circ}[(I_\circ + H_\circ)V_I(P_2,I_\circ + H_\circ) - I_\circ V_I(P_2,I_\circ] < 0 \qquad (3.52)$$

In turn a sufficient condition for inequality (3.52) to hold for the required $H_\circ$ is that it should hold for any $H_\circ$. But this is the case if:

$$\frac{\partial}{\partial I}[IV_I(P_2,I)] < 0$$

that is:

$$\frac{-IV_{II}}{V_I} > 1 \qquad (3.53)$$

Thus with homothetic preferences a sufficient condition for the higher income groups to prefer the smog free zone is that the income elasticity of the marginal utility of income be greater than unity. Conversely, if each of the above inequalities is reversed, it follows that with homothetic preferences a sufficient condition for the higher income groups to prefer the smoggy region is that the elasticity of marginal utility be less than unity.

We now note that this elasticity is also the coefficient of relative aversion to income uncertainty. Arrow (1971) has argued that the latter must be in the neighborhood of unity and increasing in income. Accepting this conclusion it follows that there is no clear presumption that income and environmental quality will be positively correlated. Indeed if relative risk aversion is less than unity for low incomes, and rises above unity as income increases it is possible for an equilibrium configuration with high and low income groups sharing the smog-free region and middle income groups in the smoggy region.

Of course this conclusion is very much dependent upon the underlying assumptions. Suppose that instead of entering additively, the environmental affects are multiplicative. That is, with the environment affected by an amount s, utility is:

$$u_1(x,y)u_2(s)$$

where $u_2(0) = 1$ and $u_2'(s) < 0$.

Each consumer chooses x, y and his location to maximize the utility or, equivalently, the logarithm of this utility, that is:

$$\ln u_1(x,y) - \ln u_2(s).$$

Setting $U(x,y) = \ln u_1(x,y)$ the problem becomes equivalent to the one already analysed. Therefore higher income groups will live in the smog free areas if the relative risk aversion of an individual with a utility function $\ln U_1(x,y)$ exceeds unity. Since ln (·) is a strictly concave function, this individual's relative risk aversion exceeds that of an individual with a utility function $U_1(x,y)$. Therefore the sufficient condition is weakened and the presumption that higher income individuals will live in the less environmentally affected area is strengthened.

3.9 Optimal Urban Location

In the previous section we considered some of the positive implications of intra urban environmental differences. It turns out that there are also rather puzzling normative implications, at least if one adopts the usual approach of maximizing a symmetric social welfare function. Suppose that initially all individuals have the same income. Some locate in the smog-free zone and the rest in the smoggy zone. A naive view might be that those living in the smog should be compensated by an income transfer from those in the smog free zone. Not so, an economist would almost certainly respond. If individuals are free to move from one zone to the other, land values will adjust to equalize utilities.

While the response is correct as far as it goes, it does not necessarily follow that the sum of all the utilities, or indeed any symmetric function of each utility, is maximized as a result. For expositional ease we shall consider only the Benthamite welfare function. Let a_i be the are of zone i, n_i the number assigned to this zone, $\bar{n}$ the total population and $\bar{y}$ the total income. We seek to maximize the utility sum:

$$W = \sum_{i=i}^{2} n_i [U(\frac{a_i}{n_i}, y_i) - s_i]$$

subject to the constraints:

$$n_1 + n_2 = \bar{n};\ n_1 y_1 + n_2 y_2 = \bar{y}$$

To solve we form a Lagrangian

$$L = W + \lambda(\bar{n} - n_1 - n_2) + \mu(\bar{y} - n_1 y_1 - n_2 y_2)$$

Necessary conditions for a maximum are therefore,

$$\frac{\partial L}{\partial y_i} = n_i (U_{y_i} - \mu) = 0. \qquad (3.54)$$

and

$$\frac{\partial L}{\partial n_i} = U(x_i, y_i) - s_i - x_i U_{x_i} - \lambda - \mu y_i = 0 \qquad (3.55)$$

where $x_i = a_i/n_i$.

Suppose that the optimal distribution of land and individuals is $(x(s_i), y(s_i))$ i = 1,2

Differentiating the two first order conditions with respect to s we have:

$$\frac{d}{ds}(U_y) = 0$$

and

$$U_x x'(s) + U_y y'(s) - 1 - x'(s)\,U_x - x\frac{d}{ds}(U_x) - \mu\frac{dy}{ds} = 0$$

Substituting for μ from (3.54) this reduces to:

$$\frac{d}{ds}(U_x) = -\frac{1}{x} \tag{3.57}$$

Writing out the derivatives in (3.56) and (3.57) we therefore have,

$$\begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix} = -\frac{1}{x}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Applying Cramer's rule yields:

$$x'(s) = -\frac{1}{x}\frac{U_{yy}}{|H_u|} \text{ and } y'(s) = -\frac{1}{x}\frac{U_{xy}}{|H_u|} \tag{3.58}$$

where H_u is the Hessian matrix of the function U(x,y). Given the concavity of U the principal minors of H_u must alternate in sign thus x'(s) > 0. It follows that the optimal plot size is larger for those located in the smoggy zone.

Furthermore, substituting from (3.58) we also have:

$$\begin{aligned}\frac{dU}{ds} &= x'(s)U_x + y'(s)U_y \\ &= -\frac{1}{x}\frac{(U_xU_{yy} - U_yU_{xy})}{|H_u|} \\ &= -\frac{1}{x}\frac{\begin{vmatrix} U_x & U_{xy} \\ U_y & U_{yy} \end{vmatrix}}{|H_u|}\end{aligned} \tag{3.59}$$

Consider an individual located in zone i facing a land price of P_i, and having an income of I. Given that he is to remain in this zone, he chooses a consumption (x_i, y_i) yielding the solution of:

$$\text{Max}\{U(x_i,y_i)\,|\,P_ix_i + y_i = I\}$$

Introducing the Lagrangian λ (equal to the marginal utility of income) the following first order conditions must be satisfied:

$$U_x = \lambda p_i$$

$$U_y = \lambda$$

Suppose income I were increased. Differentiating the first order conditions we have:

$$\begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} x_i'(I) \\ y_i'(I) \end{bmatrix} = \lambda'(I) \begin{bmatrix} P_i \\ 1 \end{bmatrix} = \frac{\lambda'(I)}{\lambda(I)} \begin{bmatrix} U_x \\ U_y \end{bmatrix}$$

Then applying Cramer's rule:

$$\frac{dx_i}{dI} = \frac{1}{\lambda}\frac{d\lambda}{dI} \frac{\begin{vmatrix} U_x & U_{xy} \\ U_y & U_{yy} \end{vmatrix}}{|H_u|} \tag{3.60}$$

Combining (3.59) and (3.60) we have:

$$\frac{dU}{ds} = \frac{-\frac{1}{x}\frac{dx_i}{dI}}{\frac{1}{\lambda}\frac{d\lambda}{dI}} = \frac{E(x_i, I)}{E(\lambda, I)}$$

The expected utility of an individual residing in zone i is $U(x_i, y_i) - s_i$. Therefore the change in expected utility as the smog level s increases is

$$\frac{dU}{ds} - 1 = \frac{E(x_i, I)}{E(\lambda, I)} - 1 \tag{3.61}$$

Therefore if the right hand side is positive for any price P_i and income level I, it is optimal for those in the smoggy zone to have a higher utility. Conversely, if the right hand side is always negative it is optimal to transfer income to those in the less smoggy zone!

For the special case of homothetic preferences examined in the previous section $(x_i, I) = 1$. Therefore in such cases it is optimal to transfer income to those in the less smoggy zone if and only if the income elasticity of marginal utility exceeds unity. Thus the condition obtained in section 2.2 ensuring that the higher income groups will locate in the less smoggy zone also ensures that for a population with equal incomes, the utility sum is maximized with a transfer of income to those in the less smoggy zone!

Such paradoxical results have already been noted in the urban literature by Mirrlees (1972) Riley (1974) and others, although the usual emphasis has been on the implications of differential transportation costs. Recently Arnott and Riley (1977) have attempted to explain the origin of these results as a production asymmetry. While their analysis does not carry over directly, to this more complicated case the basic issues are the same.

Suppose we begin with incomes equally distributed, as in Figure 3.2. Since land is cheaper in the smoggy zone plot sizes are larger, unless land is a Giffen good. That is, C_1 lies to the right of C_2. Moreover, if land is a normal good C_1' is above and to the right of C_2. Arnott and Riley note that for a normal good the marginal utility of income rises with a Hicks compensated fall in the price of the good. That is, the marginal utility of income rises around the curve from C_1' to C_1. With diminishing marginal utility of income marginal utility falls in moving from C_2 to C_1'. If the latter effect outweighs the former (and this will be the case with a sufficiently high income elasticity of marginal utility) marginal utility is lower at C_1 than at C_2. Maximization of any differentiable symmetric social welfare function therefore requires a transfer of income from those in the low marginal utility, smoggy zone to those in the less smoggy zone.

3.10 Uncertain Environmental Quality and the Prospect of Better Information

In the previous two sections we analysed the implications of environmental quality differences for property values and locational choice. Given the simple formulation of the model, none of the results are changed if s is reinterpreted as the expected utility loss associated with a polluted environment. We now consider the implications for property values of conducting research which would resolve the uncertainty about the hazards of the pollution. For expositional ease we consider the case in which the polluted region is small relative to the unpolluted region. Then to a first approximation land value and hence utility in the latter is unaffected by such information. Continuing with our assumption of a perfectly elastic response to any utility differential, it follows that expected utility in the two regions will be fixed at some level $\bar{U}$. Then prior to any consideration of research resolving uncertainty about the environmental hazard, the consumption bundle in the "rest of the world" C_0 and in the affected region C_1 yield the same expected utility level. This is depicted in Figure 3.4. Now suppose it is announced that research will reveal the true level of s. For simplicity suppose this takes one of two values s_0 (=0) and s_1. If s = 0 the utility level of individuals in regions 1 rises to $\bar{U} + E(s)$. This attracts individuals into the region and the price of land is bid up. Eventually the price of land reaches PO and outsiders no longer gain from relocation. Similarly, if $s = s_1$ the utility of those in region 1 is $\bar{U} + E(s) - s_1 < \bar{U}$. Individuals therefore leave until the price of land falls to the point where the utility differential is eliminated. Assuming individuals own their own homes, those remaining in region 1 have ex-post budget constraints:

$$P_1(s)x + y = P_1(s)x_1 + y_1$$

Final consumption is therefore dependent upon the true state s. This is also depicted in Figure 3.4. Note that in both states we have:

$$U(C_1(s)) > U(C_1)$$

In anticipation of the release of the information about s, expected utility in region 1 is therefore:

$$\underset{s}{E}(U(C_1(s)) - s) = \underset{s}{E}U(C_1(s)) - E(s) > U(C_1) - E(s) = \bar{U}$$

Figure 3.4

Uncertainty and Better Information

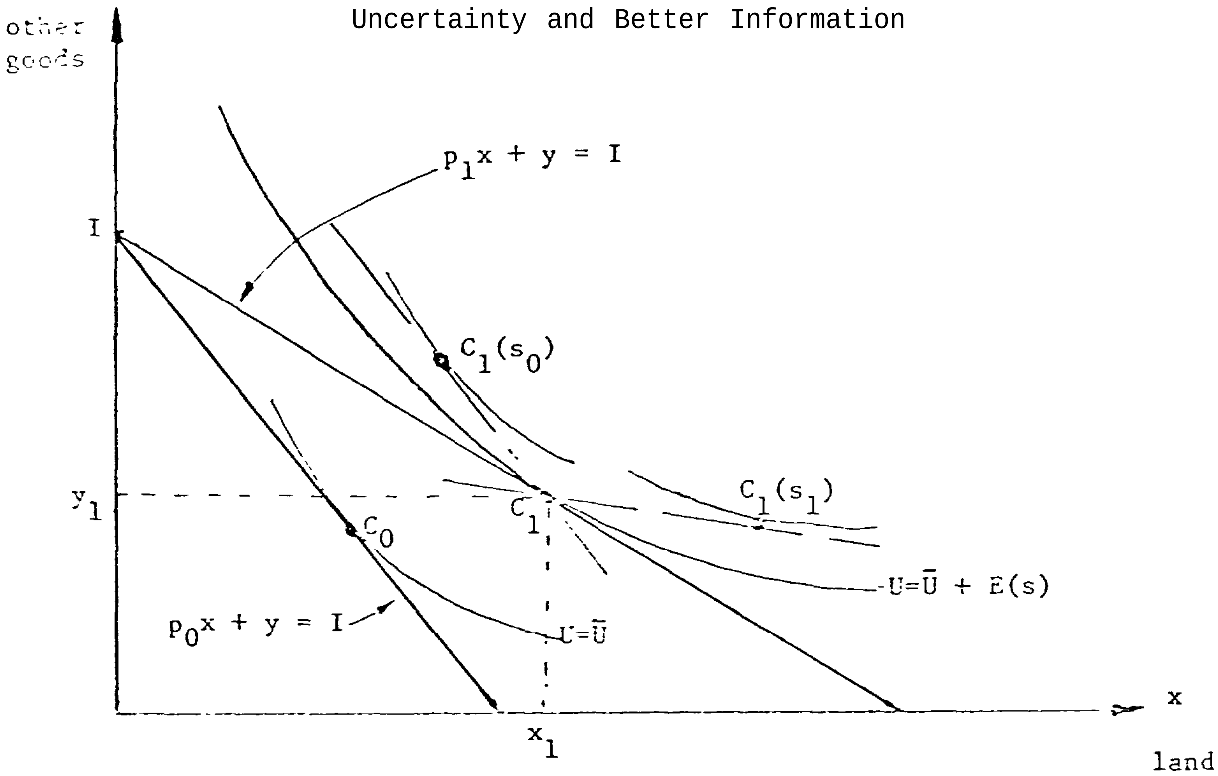
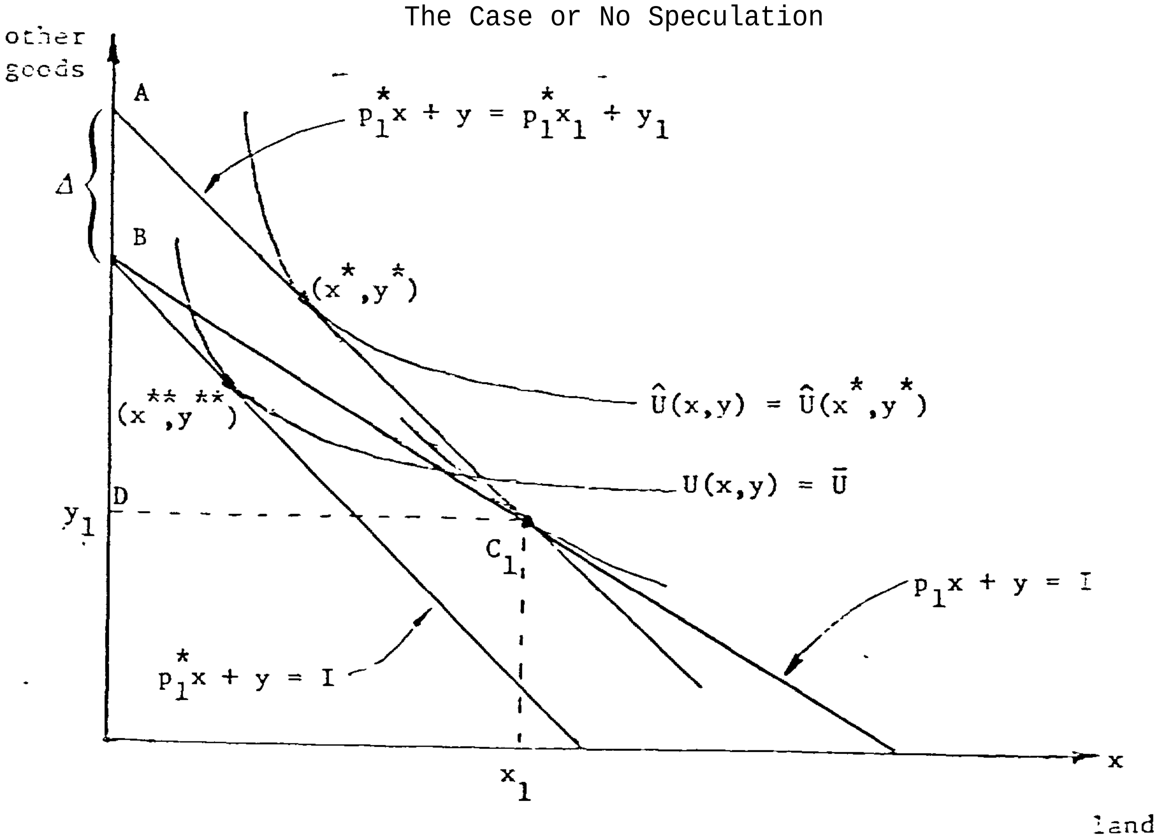


Figure 3.5

The Case or No Speculation



Therefore all homeowners in region 1 are made strictly better off by the announcement of the proposed research. As a result outsiders will wish to relocate in region 1. The value of land is therefore bid up to some level p^* where the expected utility achieved by relocation once again falls to $\overline{U}$.

The budget constrainst of those initially in region 1 and those moving into the region are depicted in Figure 3.5 under the assumption that the price of land jumps too quickly for significant speculative activity.

Suppose the former group chooses a bundle (x^*,y^*) and the latter (x^{**}, y^{**}). Each group of course anticipates retrading at a later point. Since both face an expected loss due to the environmental hazard of E(s) we can write the utility differential as:

$$\hat{U}(x^*,y^*) - \hat{U}(x^{**},y^{**})$$

where $\hat{U}(x,y) \underset{P_1}{-} EV(p_1,p_1x + y)$ is the derived utility function for both groups.

Of course there is no simple relationship between the indifference curves for the derived utility function U(x,y) and the underlying function U(x,y). However it must be the case that those entering the region have the expected utility level $\overline{U}$. That is: $U(x^{**},y^{**}) = \overline{U}$.

It follows that $\hat{U}(x^*,y^*) - \hat{U}(x^{**},y^{**})$ is the gain in expected utility for those located initially in region 1. Consider again Figure 3.5. In order for those entering region 1 to achieve as high a utility level as the initial land owners, it would be necessary to increase the income of each from I to $I+\Delta$. Thus Δ is a measure of the dollar valuation of the information. Note that $AD=p_1^* x_1$ and $BD=p_1x_1$. Therefore the value of information to each individual initially located in region 1 is:

$$\Delta = (P_1^* - P_1)x_1$$

Aggregating over the whole region, the total value of the information is equal to the increase in the value of the land in the region.

Unfortunately it is difficult to visualize how one might make a quantitative prediction of the extent of this revaluation without working back to the underlying preferences. In a later draft we intend to illustrate how this might be done for the Cobb-Douglas case.

3.11 Precautionary Response to the Prospect of Information

Section 3.1 explores the value to an individual of receiving either perfect or partial information about product quality prior to making <u>any</u> consumption decisions. Consumption decisions were binding once made and could not be altered if subsequent information about s arrived. It is generally the case, however, that once an individual (or society) does choose to acquire additional information about some good it takes some time to produce it through experimentation and research. In the meantime current consumption decisions must still be made, although future consumption plans may be appropriately revised upon receipt of the experimental